TWISTED CONJUGACY IN FREE GROUPS AND MAKANIN'S QUESTION

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Abstract. We discuss the following question of G. Makanin from "Kourovka notebook": does there exist an algorithm to determine is for an arbitrary pair of words U and V of a free group F_n and an arbitrary automorphism $\varphi \in \mathrm{Aut}(F_n)$ the equation $\varphi(X)U = VX$ solvable in F_n? We give the affirmative answer in the case when an automorphism is virtually inner, i.e. some its non-zero power is an inner automorphism of F_n.

1. Conjugacy and twisted conjugacy

Suppose G is a group given by a presentation in generators and defining relations. Three following decision problems formulated by M. Dehn [6] in 1912 (see also [9, Ch. 1, § 2; Ch. 2, § 1]) are fundamental in the group theory.

Word problem: *Does there exist an algorithm to determine if an arbitrary group word W given in the generators of G defines the identity element of G?*

Conjugacy problem: *Does there exist an algorithm to determine is an arbitrary pair of group words U, V in the generators of G define conjugate elements of G?*

Isomorphism problem: *Does there exist an algorithm to determine for any two arbitrary finite presentations whether the groups they present are isomorphic or not?*

All three of these problems have negative answers in general (see, for example [1], [3, Ch. 6.7]). These results together with solutions of Dehn's problems in restricted cases have been of central importance in the combinatorial group theory. For this reason combinatorial group theory has always searched for and studied classes of groups in which these decision problems are solvable. Also, various generalizations of these problems were considered (see, for example [7], [9, Ch. 4, § 4]).

In the present paper we discuss the twisted conjugacy problem, that can be considered as a generalization of Dehn's conjugacy problem. Suppose G

Date: February 1, 2008.

1991 *Mathematics Subject Classification.* 20F10.

Key words and phrases. Conjugacy problem, free groups, automorphism.

Authors were supported in part by the Russian Foundation for Basic Research (grants 01–01–00630 and 02–01–01118).

is a group given by a presentation and H is a subset of its automorphisms group $\mathrm{Aut}(G)$.

Twisted conjugacy problem: *Does there exist an algorithm to determine is for an arbitrary pair of group words U and V in generators of G the equality $\varphi(W)U = VW$ holds for some $W \in G$ and $\varphi \in H$?*

Since the twisted conjugacy problem depends of a group G as well as a subset $H \subseteq \mathrm{Aut}(G)$, this problem will be referred to as the conjugacy problem in G *relative to* H or as the H*–twisted conjugacy problem in* G. If H consists of the identity automorphism then we deal with the classical conjugacy problem. If $H = \{\varphi\}$ consists of the unique automorphism then, obviously, a property to be twisted conjugated is an equivalence relation.

The following question was posted by G. Makanin in "Kourovka notebook" [8, Question 10.26(a)]: *Does there exist an algorithm to determine is for an arbitrary pair of words U and V of a free group G and an arbitrary automorphism φ of G the equation $\varphi(X)U = VX$ solvable in G?* This question can be regarded as the twisted conjugacy problem for the case when G is a free group and $H = \{\varphi\}$ consists of the unique automorphism.

We remark that the twisted conjugacy problem is also connected with the question of solving equations in the holomorph $\mathrm{Hol}(G) = G \ltimes \mathrm{Aut}(G)$. In this context we recall the following question of G. Makanin [8, Question 10.26(b)]: *Does there exist an algorithm to determine is for arbitrary automorphisms $\varphi_1, \dots, \varphi_n$ of a free group the equation $w(x_{i_1}^{\varphi_1}, \dots, x_{i_n}^{\varphi_n}) = 1$ solvable?*

2. Basic notations and results

In the present paper we deal with the case when $G = F_n$ is the finitely generated free group with basis $\{x_1, \dots, x_n\}$. Elements of F_n are words in the alphabet $\mathbb{X} = \{x_1^{\pm 1}, \dots, x_n^{\pm 1}\}$. In our considerations below we will need to distinguish words which represent the same element of the group. So, we will usually use small letters to denote elements of F_n and capital letters to denote words in the above alphabet. We will write $U = V$ (or $u = v$) if two words (or two elements of the group) are equal as elements of the group, and $U \equiv V$ if two words coincide graphically. We use standard notations $\mathrm{Aut}(F_n)$ and $\mathrm{Inn}(F_n)$ for the group of automorphisms and the group of inner automorphisms of F_n.

Denote by $|W|$ the length of a word W in the alphabet $\mathbb{X}$. A word W is said to be *reduced* if it contains no part xx^{-1}, $x \in \mathbb{X}$. A reduced word W defines a non-identity element if and only if $|W| \geq 1$. A reduced word obtained by reducing of an original word will be referred to as its *reduction*. By $||W||$ we will denote the length of the reduction of a word W.

We define a linear order "$<$" on the set of reduced words in the alphabet $\mathbb{X}$. Assume that letters of $\mathbb{X}$ are ordered in the following way:

$$x_1 < x_1^{-1} < x_2 < x_2^{-1} < \dots < x_n < x_n^{-1}.$$

We write $U < V$ if $|U| < |V|$ or if $|U| = |V|$ and the word U is less than the word V in respect to the lexicographical order corresponding to the above defined linear order on $\mathbb{X}$.

Recall that all three classical Dehn problems have simple and elegant solutions in free groups. Since reducing any word is an algorithmic process, this provides a solution of the word problem. Further, a reduced word $W = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_n}^{\varepsilon_n}$, where $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$, is said to be *cyclically reduced* if $i_1 \neq i_n$ or if $i_1 = i_n$ then $\varepsilon_1 \neq -\varepsilon_n$. Clearly, every element of a free group is conjugated to an element given by a cyclically reduced word called a *cyclic reduction*. This leads to a solution of the conjugacy problem. Suppose for given words U and V, words $\overline{U}$, $\overline{V}$ are their cyclic reductions. Then U is conjugated to V if and only if $\overline{U}$ is a cyclic permutation of $\overline{V}$. Finally, two finitely generated free groups are isomorphic if and only if they have the same rank.

In this paper we consider the twisted conjugacy problem in $G = F_n$ relative to $H = \{\varphi\}$, where $\varphi \in \mathrm{Aut}(F_n)$ is such that $\varphi^m \in \mathrm{Inn}(F_n)$ for some non-zero integer m. Such automorphism φ will be referred to as a *virtually inner automorphism*. In particular, if an automorphism φ is of finite order or inner, then it is virtually inner. The main result of the paper is the following

Theorem 1. *Suppose φ is a virtually inner automorphism of the free group F_n. Then the φ–twisted conjugacy problem in F_n is solvable.*

To prove this statement, we introduce φ-twisted conjugated normal form of a word, which can be constructed by a finite number of steps. We show that two words are φ-twisted conjugated if and only if their φ-twisted conjugated normal form coincide (see Section 6 for the proof).

Consider the mapping torus (in other words, the ascending HNN extension)

$$F_n(\varphi) = \langle x_1, \ldots, x_n, t \mid t^{-1} x_i t = \varphi(x_i),\ i = 1, \ldots, n \rangle$$

that is the semidirect product of F_n and $\langle t \rangle$. There is an evident relation between the φ-twisted conjugacy problem in F_n and the classical conjugacy problem in $F_n(\varphi)$.

Lemma 1. *The φ–twisted conjugacy problem in the free group F_n is solvable if and only if there exists an algorithm to determine is a pair of words tU and tV from $F_n(\varphi)$, where $U, V \in F_n$, define elements conjugated by some element of F_n.*

Proof. Obviously, the equality $X^{-1}(tV)X = tU$ holds in $F_n(\varphi)$ for some $X \in F_n$ if and only if the equality $\varphi(X^{-1})VX = U$ holds in F_n, i.e. $\varphi(X)U = VX$. □

It was shown by Bestvina and Feighn in [2], that if $\varphi \in \mathrm{Aut}(F_n)$ have no nontrivial periodic conjugacy classes (see next section for the definition) then $F_n(\varphi)$ is hyperbolic. So, the conjugacy problem is solvable in it. (We

note, that virtually inner automorphisms that we consider below, are such that each conjugacy class is periodic.) We will show in Proposition 1 that also the problem of conjugation by an element of F_n is solvable in it.

Theorem 1, Lemma 1, and Proposition 1 imply the partial affirmative answer on the above mentioned Makanin's question [8, Question 10.26(a)]:

Corollary 1. *Let $\varphi \in Aut(F_n)$ be a virtually inner or having no nontrivial periodic conjugacy classes. Then there exists an algorithm to determine for an arbitrary pair of words U and V the solvability in F_n of the equation $\varphi(X)U = VX$.*

3. Automorphisms having no nontrivial periodic conjugacy classes

An automorphism φ of the free group F_n is said to have *nontrivial periodic conjugacy class* if there exist integer ℓ and elements $x, y \in F_n$ such that $\varphi^\ell(x) = y^{-1}xy$. It was proven by Bestvina and Feighn [2] and by P. Brinkmann [5], that $F_n(\varphi)$ is hyperbolic (in the Gromov's sense) if and only if automorphism φ has no nontrivial periodic conjugacy classes.

Proposition 1. *Suppose $\varphi \in Aut(F_n)$ has no nontrivial periodic conjugacy classes. Then the φ-twisted conjugacy problem in F_n is solvable.*

Proof. By Lemma 1, the φ–twisted conjugacy of words U and V from F_n is equivalent to the conjugacy of words tU and tV from $F_n(\varphi)$ by some element of F_n. Since φ has no nontrivial periodic conjugacy classes, by [2] the group $F_n(\varphi)$ is hyperbolic and so, the conjugacy problem in this group is solvable (see, for example, [4, Ch. 3]).

If tU and tV are not conjugated in $F_n(\varphi)$, then they, in particular, are not conjugated by an element of F_n, so U and V are not φ–twisted conjugated.

Assume that tU and tV are conjugated in $F_n(\varphi)$ by some element t^kW, where $W \in F_n$ and k is integer. If $k = 0$, the statement follows from Lemma 1. If $k \neq 0$, we remark the following. Each element of $F_n(\varphi)$ that conjugates tU to tV is of the form $c \cdot t^kW$, where c belongs to the centralizer $C_G(tU)$ of tU in $G = F_n(\varphi)$. It was shown by Gromov (see, for example, [4, Corollary 3.10, p. 462]), that in a hyperbolic group the centralizer of any nontrivial element is almost cyclic and there exists an algorithm to find generators of the centralizer $C_G(tU)$. Considering the canonical homomorphism of the centralizer $C_G(tU)$ onto the group $\langle t \rangle$, one can check if the element t^{-k} lies in the image of that homomorphism. If yes, then elements tU and tV are conjugated by an element of F_n. If not, then they are not conjugated by an element of F_n, and so, elements U and V are not φ-twisted conjugated. □

We are thankful to Oleg Bogopol'skii for useful discussion of results from this Section.

4. Virtually inner automorphisms of free groups

Let $\varphi \in \mathrm{Aut}(F_n)$ be a virtually inner automorphism such that $\psi = \varphi^m$ is an inner automorphism of F_n. Without loss of generality we can assume that m is taken the smallest positive integer having such a property. There exists a reduced word $\Delta \in F_n$ such that

$$\varphi^m(f) \,=\, \Delta^{-1} f \Delta$$

for any $f \in F_n$.

Lemma 2. *(1) Automorphisms φ and ψ commute.*
(2) If $k = mq + r$, where $q \in \mathbb{Z}$ and $0 \le r \le m-1$, then for any word U of the alphabet $\mathbb{X}$ we have

$$\varphi^k(U) \,=\, \varphi^r(\Delta^{-q}\, U\, \Delta^q).$$

(3) $\varphi(\Delta) = \Delta$.

Proof. Properties (1) and (2) hold obviously, since ψ is a power of φ.
To show (3), remark that by (1) we have

$$\Delta^{-1}\,\varphi(f)\,\Delta \,=\, \varphi(\Delta^{-1} f \Delta) \,=\, \varphi(\Delta^{-1})\,\varphi(f)\,\varphi(\Delta)$$

for any $f \in F_n$. Therefore, $\varphi(\Delta)\,\Delta^{-1}$ and $\varphi(f)$ commute for all $f \in F_n$. Since φ is automorphism, we get $\varphi(\Delta)\,\Delta^{-1} = 1$, so $\varphi(\Delta) = \Delta$. $\square$

A reduced word $V \in F_n$ is said to be Δ-*reduced* if $|V| \le ||\Delta^{-k} V \Delta^k||$ for all $k \in \mathbb{Z}$. Obviously, if V is cyclically reduced, then the length of any word conjugated to V is not less than the length of V, so V is Δ-reduced.

Lemma 3. *Suppose that Δ is cyclically reduced. A reduced word $V \in F_n$ is Δ-reduced if $|V| \le ||\Delta^{-\varepsilon} V \Delta^{\varepsilon}||$ for $\varepsilon = \pm 1$.*

Proof. If V is cyclically reduced, by the above observation it is Δ-reduced. If V is not cyclically reduced, we represent it in the form $V \equiv U^{-1} W U$, where U and W are reduced nonempty words in F_n and W is cyclically reduced.

Suppose that $|V| \le ||\Delta^{-1} V \Delta||$ and assume that there exists integer $k > 1$ such that $||\Delta^{-k} V \Delta^k|| < |V|$. Then

$$\Delta^{-k}\, V\, \Delta^k \,\equiv\, \Delta^{-k}\, U^{-1}\, W\, U\, \Delta^k \,\equiv\, (U \Delta^k)^{-1}\, W\, (U \Delta^k).$$

Since Δ is cyclically reduced, $||\Delta^k|| = k\,|\Delta|$ and in the word $U\Delta^k$ only cancellations of letters from U with letters from Δ^k can arise. If such cancellations are possible, there are two possibilities: either there are cancellations with letters of W or not.

Case 1. Suppose that there are cancellations of letters of Δ^k with letters of U, where, possibly, U will be cancelled wholly (the same for Δ^{-k} and U^{-1}, respectively), but there no further cancellations with letters of W. Then we can write $U \equiv \Sigma_1 \Sigma_2^{-1}$ and $\Delta^k \equiv \Sigma_2 \Sigma_3$ for some reduced $\Sigma_1, \Sigma_2, \Sigma_3 \in F_n$.

If the length of the cancelling part Σ_2 is bigger than the length of Δ, then $\Sigma_2 \equiv \Delta\,\Sigma_{2,1}$ and $U \equiv \Sigma_1\,\Sigma_{2,1}^{-1}\,\Delta^{-1} \equiv U_1\,\Delta^{-1}$, where $|U_1| < |U|$. Then

$$||\Delta^{-1}\,V\,\Delta|| = ||\Delta^{-1}U^{-1}WU\Delta|| = ||U_1^{-1}WU_1|| = |W| + 2|U_1|$$

that is less than

$$|V| = |U^{-1}WU| = |W| + 2|U|$$

because $|U_1| < |U|$. Thus we get the contradiction with the assumption.

If the length of the cancelling part Σ_2 is less or equal to the length of Δ, then $U \equiv \Sigma_1\Sigma_2^{-1}$ and $\Delta^k \equiv \Sigma_2\Sigma_3\Delta^{k-1}$ for some reduced $\Sigma_1, \Sigma_2, \Sigma_3 \in F_n$. We get

$$|V| = |W| + 2|U| = |W| + 2|\Sigma_1| + 2|\Sigma_2|,$$
$$||\Delta^{-1}V\Delta|| = ||\Sigma_3^{-1}\Sigma_1^{-1}W\Sigma_1\Sigma_3|| = |W| + 2|\Sigma_1| + 2|\Sigma_3|,$$

and

$$||\Delta^{-k}V\Delta^k|| = ||\Delta^{-(k-1)}(\Delta^{-1}V\Delta)\Delta^{k-1}||$$
$$= ||\Delta^{-1}V\Delta|| + 2(k-1)|\Delta| = |W| + 2|\Sigma_1| + 2|\Sigma_3| + 2(k-1)|\Delta|.$$

Since $k > 1$ and $|\Sigma_2| \leq |\Delta|$, we see that $|V| \leq ||\Delta^{-k}V\Delta^k||$ that gives the contradiction with the assumption.

Case 2. Suppose that letters of Δ^{-k} and Δ^k are cancelling with letters of $V \equiv U^{-1}WU$ is such a way that words U and U^{-1} will be cancelled wholly and there are further cancellations with letters of W. Since $|U^{-1}WU| \leq ||\Delta^{-1}U^{-1}WU\Delta||$, and U^{-1} and U are cancelling wholly, we have $U \equiv \Delta_1^{-1}$ and $\Delta \equiv \Delta_1\,\Delta_2$ with $|\Delta_2| \geq |\Delta_1| = |U|$. Hence

$$\begin{array}{rcl} \Delta^{-k}\,V\,\Delta^k & \equiv & \Delta^{-(k-1)}\,\Delta_2^{-1}\,\Delta_1^{-1}\,(\Delta_1\,W\,\Delta_1^{-1})\,\Delta_1\,\Delta_2\,\Delta^{k-1} \\ & = & \Delta^{-(k-1)}\,\Delta_2^{-1}\,W\,\Delta_2\,\Delta^{k-1}. \end{array}$$

The further cancellations are possible either in $\Delta_2^{-1}\,W$ or in $W\Delta_2$, but not in both, since W is cyclically reduced.

If Δ_2^{-1} is cancelling wholly in $\Delta_2^{-1}W\Delta_2$, then

$$||\Delta^{-1}V\Delta|| = ||\Delta_2^{-1}W\Delta_2|| = |W| < |V|$$

and we have a contradiction with the assumption $||\Delta^{-1}V\Delta|| \geq |V|$.

If Δ_2^{-1} is not cancelling wholly in the product $\Delta_2^{-1}W\Delta_2$, then $\Delta_2 \equiv \Delta_{21}\Delta_{22}$ and $W \equiv \Delta_{21}^{-1}W_1$, and

$$\begin{array}{rcl} ||\Delta^{-k}V\Delta^k|| & = & ||\Delta^{-(k-1)}\Delta_2^{-1}W\Delta_2\Delta^{k-1}|| \\ & = & ||\Delta^{-(k-1)}\Delta_{22}^{-1}W_1\Delta_{21}\Delta_{22}\Delta^{k-1}|| \\ & \geq & |W| + 2(k-1)\,|\Delta| + 2|\Delta_{22}| \\ & > & |W| + 2\,|\Delta| \; > \; |W| + 2|U| = |V|, \end{array}$$

and we have a contradiction with the assumption $||\Delta^{-k}V\Delta^k|| < |V|$. By similar arguments, the case when Δ_2 is not cancelling wholly in the product $W\Delta_2$, is also impossible.

If there exists integer $k < -1$ with the same property, similar considerations implies the statement. $\square$

If Δ is cyclically reduced, Lemma 3 gives the finite algorithm to find for a given reduced word V a Δ-reduced word V_Δ conjugated to V by some power of Δ. Indeed, it is enough to repeat conjugations of V by Δ^ε, $\varepsilon = \pm 1$, few times. If the length of the obtained word is less than the length of the previous word, we will conjugate again. If not, then the obtained word is a Δ-reduced word V_Δ conjugated to V. Such a construction of a Δ-reduced word V_Δ conjugated to V by some power of Δ will be referred to as a Δ-*reduction.*

We remark that if Δ is not cyclically reduced, then the analog of Lemma 3 does not hold. It is clear from the following example.

Example. Let $U, W, \Sigma \in F_n$ be nonempty reduced words such that for an integer $|k| > 1$ words $\Delta \equiv U^{-1}W^{-1}\Sigma W U$ and $V \equiv U^{-1}W^{-1}\Sigma^k W U^2$ are reduced. If $k > 1$, we get

$$\begin{array}{rcl}\Delta^{-1}V\Delta & = & U^{-1}W^{-1}\Sigma^{k-1}WUW^{-1}\Sigma WU,\\ \Delta^{-2}V\Delta^2 & = & U^{-1}W^{-1}\Sigma^{k-2}WUW^{-1}\Sigma^2 WU,\\ & \dots & \\ \Delta^{-(k-1)}V\Delta^{k-1} & = & U^{-1}W^{-1}\Sigma WUW^{-1}\Sigma^{k-1}WU,\\ \Delta^{-k}V\Delta^k & = & W^{-1}\Sigma^k WU.\end{array}$$

It is easy to see that

$$|V| < ||\Delta^{-1}V\Delta|| = ||\Delta^{-2}V\Delta^2|| = \ldots = ||\Delta^{-(k-1)}V\Delta^{k-1}||,$$

but $||\Delta^{-k}V\Delta^k|| < |V|$.

If $k < -1$, similar example can be obtained conjugating V by negative powers of Δ.

Lemma 4. *Suppose Δ is not cyclically reduced. Let V be a reduced word such that $|V| \le ||\Delta^{-\varepsilon}V\Delta^\varepsilon||$, $\varepsilon = \pm 1$. Then one of the following cases holds:*
(1) V is Δ-reduced.
(2) $\Delta \equiv U_1^{-1}U_2^{-1}\Delta_{11}^{-1}\Delta_2\Delta_{11}U_2U_1$ and $V \equiv U_1^{-1}U_2^{-1}WU_2U_1$, for some reduced $\Delta_{11}, \Delta_2, U_1, U_2$ and cyclically reduced W for which there exist integers k, $|k| > 1$, and $m \ge 0$ and reduced Φ, W_0 such that either
a) $U_2^{-1}\Delta_{11}^{-1}\Delta_2^{-k}\Delta_{11} \equiv \Phi W^{-m}$ and $W \equiv \Phi^{-1}W_0$,
or
b) $\Delta_{11}^{-1}\Delta_2^k\Delta_{11}U_2 \equiv W^{-m}\Phi$ and $W \equiv W_0\Phi^{-1}$.
In addition, Φ does not end by W^{-1}, and W_0 is either nonempty or $W_0 \equiv 1$ with $U_1^{-1}\Phi^{-1}U_1$ be reduced.

In case (2) $V_\Delta = \Delta^{-k}V\Delta^k$ is Δ-reduced word conjugated to V. Moreover, case (2) describe all possible cases when $|V| \le ||\Delta^{-\varepsilon}V\Delta^\varepsilon||$, $\varepsilon = \pm 1$, but $||\Delta^{-k}V\Delta^k|| < |V|$ for some $|k| > 1$.

Proof. Suppose that $|V| \le ||\Delta^{-1}V\Delta||$ and assume that there exists integer $k > 1$ such that $||\Delta^{-k}V\Delta^k|| < |V|$.

We represent the reduced word Δ in the form $\Delta \equiv \Delta_1^{-1}\Delta_2\Delta_1$, where Δ_2 is cyclically reduced and Δ_1 is reduced. Then $\Delta^k = (\Delta_1^{-1}\Delta_2\Delta_1)^k = \Delta_1^{-1}\Delta_2^k\Delta_1$, where the last word is reduced and $||\Delta^k|| = 2|\Delta_1| + k|\Delta_2|$.

If V is cyclically reduced, then, obviously, it is Δ-reduced.

If V is not cyclically reduced, we write it in the form $V \equiv U^{-1}WU$, where W is cyclically reduced and U is reduced. Then

$$\Delta^{-k} V \Delta^{k} = \Delta^{-k} U^{-1} W U \Delta^{k} = (U \Delta_1^{-1} \Delta_2^{k} \Delta_1)^{-1} W (U \Delta_1^{-1} \Delta_2^{k} \Delta_1).$$

Since $||\Delta^{k} V \Delta^{-k}|| < |V|$, there are cancellations in this word. There are two possibilities: either there are cancellations with letters of W or not.

<u>Case 1.</u> Suppose that there are cancellations of letters of Δ^k with letters of U, where, possibly, U will be cancelled wholly (the same for Δ^{-k} and U^{-1}, respectively), but there no further cancellations with letters of W. Then we can write $U \equiv \Sigma_1 \Sigma_2^{-1}$ and $\Delta_1^{-1} \Delta_2^{k} \Delta_1 \equiv \Sigma_2 \Sigma_3$ for some $\Sigma_1, \Sigma_2, \Sigma_3 \in F_n$.

Since

$$||\Delta^{-k} V \Delta^{k}|| = |\Sigma_3^{-1} \Sigma_1^{-1} W \Sigma_1 \Sigma_3| = |W| + 2|\Sigma_1| + 2|\Sigma_3|$$

and

$$|V| = |\Sigma_2 \Sigma_1^{-1} W \Sigma_1 \Sigma_2^{-1}| = |W| + 2|\Sigma_1| + 2|\Sigma_2|,$$

we get $|\Sigma_2| > |\Sigma_3|$.

From the representation $\Delta_1^{-1} \Delta_2^{k} \Delta_1 \equiv \Sigma_2 \Sigma_3$ we have $\Sigma_2 \equiv \Delta_1^{-1} \Delta_2^{\ell} \Delta_{21}$ and $\Sigma_3 \equiv \Delta_{22} \Delta_2^{m} \Delta_1$, where $\Delta_2 \equiv \Delta_{21} \Delta_{22}$ (one of words Δ_{21} and Δ_{22} can be empty) and $k = \ell + m + 1$.

Consider

$$\begin{aligned} ||\Delta^{-1} V \Delta|| &= ||\Delta_1^{-1} \Delta_2^{-1} \Delta_1 U^{-1} W U \Delta_1^{-1} \Delta_2 \Delta_1|| \\ &= ||\Delta_1^{-1} \Delta_2^{-1} \Delta_1 \Sigma_2 \Sigma_1^{-1} W \Sigma_1 \Sigma_2^{-1} \Delta_1^{-1} \Delta_2 \Delta_1|| \\ &= ||\Delta_1^{-1} \Delta_2^{-1} \Delta_1 (\Delta_1^{-1} \Delta_2^{\ell} \Delta_{21}) \Sigma_1^{-1} W \Sigma_1 (\Delta_{21}^{-1} \Delta_2^{-\ell} \Delta_1) \Delta_1^{-1} \Delta_2 \Delta_1|| \\ &= ||\Delta_1^{-1} \Delta_2^{\ell-1} \Delta_{21} \Sigma_1^{-1} W \Sigma_1 \Delta_{21}^{-1} \Delta_2^{-\ell+1} \Delta_1|| \\ &= |W| + 2\,|\Sigma_1| + 2\,|\Delta_1^{-1} \Delta_2^{\ell-1} \Delta_{21}| \end{aligned}$$

and

$$|V| = |\Sigma_2 \Sigma_1^{-1} W \Sigma_1 \Sigma_2^{-1}| = |W| + 2|\Sigma_1| + 2|\Sigma_2|.$$

Since $\Sigma_2 \equiv \Delta_1^{-1} \Delta_2^{\ell} \Delta_{21}$, we have $|\Sigma_2| > |\Delta_1^{-1} \Delta_2^{\ell-1} \Delta_{21}|$. Therefore, $||\Delta^{-1} V \Delta|| < |V|$, that gives the contradiction with the assumption.

Therefore, for any $k \geq 1$ the inequality $|V| \leq ||\Delta^{-k} V \Delta^{k}||$ holds.

Taking $\Delta V \Delta^{-1}$, similar considerations show that the inequality holds also for $k \leq -1$. Therefore, V is Δ-reduced word.

<u>Case 2.</u> Suppose that letters of Δ^{-k} and Δ^{k}, where $\Delta^{k} \equiv \Delta_1^{-1} \Delta_2^{k} \Delta_1$ are cancelling with letters of $V \equiv U^{-1} W U$ is such a way that words U and U^{-1} will be cancelled wholly and there are further cancellations with letters of W, starting either from the initial part of W or from the final part of W, but not from the both, since W is cyclically reduced.

<u>Case 2(i).</u> Suppose that $|U| \geq |\Delta_1|$ and cancellations in W starts from the initial part.

Recall that $|V| \leq ||\Delta^{-1} V \Delta||$, where $|V| = |W| + 2|U|$ and $||\Delta^{-1} V \Delta|| = ||\Delta_1^{-1} \Delta_2^{-1} \Delta_1 U^{-1} W U \Delta_1^{-1} \Delta_2 \Delta_1||$. Hence $|U| \leq |\Delta_1| + \frac{1}{2}|\Delta_2|$ and we can

represent $U^{-1} \equiv \Delta_1^{-1}\Delta_{21}$, where $\Delta_2 \equiv \Delta_{21}\Delta_{22}$ and $|\Delta_{21}| \leq |\Delta_{22}|$. Then

$$\begin{aligned}\Delta^{-k}V\Delta^k &= \Delta_1^{-1}(\Delta_{21}\Delta_{22})^{-k}\Delta_1(\Delta_1^{-1}\Delta_{21})W(\Delta_{21}^{-1}\Delta_1)\Delta_1^{-1}(\Delta_{21}\Delta_{22})^k\Delta_1 \\ &= \Delta_1^{-1}(\Delta_{21}\Delta_{22})^{-(k-1)}\Delta_{22}^{-1}W\Delta_{22}(\Delta_{21}\Delta_{22})^{k-1}\Delta_1.\end{aligned}$$

Consider

$$|V| = |U^{-1}WU| = |W| + 2|U| = |W| + 2|\Delta_1| + 2|\Delta_{21}|$$

and

$$\Delta^{-1}V\Delta = \Delta_1^{-1}\Delta_2^{-1}\Delta_1U^{-1}WU\Delta_1^{-1}\Delta_2\Delta_1 = \Delta_1^{-1}\Delta_{22}^{-1}W\Delta_{22}\Delta_1.$$

Since $|V| \leq ||\Delta^{-1}V\Delta||$, in the product $\Delta_{22}^{-1}W$ the word Δ_{22}^{-1} can not be cancelled wholly.

Indeed, if Δ_{22}^{-1} is cancelling wholly, then after cancellations in $\Delta_{22}^{-1}W\Delta_{22}$ we will get a cyclically reduced word which is obtained by a cyclic shift of W. Therefore $||\Delta_{22}^{-1}W\Delta_{22}|| = |W|$ and

$$||\Delta^{-1}V\Delta|| \leq |W| + 2|\Delta_1| < |V|,$$

and we will get the contradiction.

Therefore, we can represent $\Delta_{22}^{-1} \equiv \Sigma_0\Sigma_1^{-1}$, where Σ_1^{-1} is the cancelling part and Σ_0 is non-empty. After cancellations in $\Sigma_1^{-1}W\Sigma_1$ we will get a cyclically reduced word which is obtained by a cyclic shift of W. Therefore, $||\Sigma_1^{-1}W\Sigma_1|| = |W|$ and

$$||\Delta^{-1}V\Delta|| = ||\Delta_1^{-1}\Sigma_0(\Sigma_1^{-1}W\Sigma_1)\Sigma_0^{-1}\Delta_1|| = |W| + 2|\Delta_1| + 2|\Sigma_0|.$$

Comparing with

$$\begin{aligned}||\Delta^{-k}V\Delta^k|| &= ||\Delta_1^{-1}(\Delta_{21}\Delta_{22})^{-(k-1)}\Delta_{22}^{-1}W\Delta_{22}(\Delta_{21}\Delta_{22})^{k-1}\Delta_1|| \\ &= ||\Delta_1^{-1}(\Delta_{21}\Delta_{22})^{-(k-1)}\Sigma_0(\Sigma_1^{-1}W\Sigma_1)\Sigma_0^{-1}(\Delta_{21}\Delta_{22})^{k-1}\Delta_1|| \\ &= |W| + 2|\Delta_1| + 2|\Sigma_0| + 2(k-1)|\Delta_{21}\Delta_{22}|,\end{aligned}$$

we see that $||\Delta^{-1}V\Delta|| < ||\Delta^{-k}V\Delta^k||$ for all $k > 1$. By similar considerations the inequality holds also for all $k < -1$. Therefore, V is Δ-reducible.

Case 2(ii). Suppose that $|U| \geq |\Delta_1|$ and cancellations in W starts from the final part. The arguments analogous to the arguments from the Case 2(i), shows that in this case V Δ-reducible.

Case 2(iii). Suppose that $|U| < |\Delta_1|$ and cancellations in W starts from the initial part. Then we can write $\Delta_1 \equiv \Delta_{11}U$, where $\Delta_{11} \neq 1$. Therefore,

$$\begin{aligned}\Delta^{-k}V\Delta^k &\equiv \Delta_1^{-1}\Delta_2^{-k}\Delta_1U^{-1}WU\Delta_1^{-1}\Delta_2^k\Delta_1 \\ &= U^{-1}\Delta_{11}^{-1}\Delta_2^{-k}\Delta_{11}UU^{-1}WUU^{-1}\Delta_{11}^{-1}\Delta_2^k\Delta_{11}U \\ &= U^{-1}\Delta_{11}^{-1}\Delta_2^{-k}\Delta_{11}W\Delta_{11}^{-1}\Delta_2^k\Delta_{11}U.\end{aligned}$$

Since $||\Delta^{-k}V\Delta^k|| < |V|$, where $|V| = |W| + 2|U|$, is it necessary that in the product $U^{-1}\Delta_{11}^{-1}\Delta_2^{-k}\Delta_{11}W$ the word $\Delta_{11}^{-1}\Delta_2^{-k}\Delta_{11}$ will be cancelled wholly and, moreover, there will be some cancellations of letters of U^{-1} with letters of W. Denote by U_2^{-1} the cancelling part of the word U^{-1}, Then we can

write $U \equiv U_2U_1$, where $|U| = |U_1| + |U_2|$, and $W \equiv \Delta_{11}^{-1}\Delta_2^k\Delta_{11}U_2W_0$, for some W_0. Since W is cyclically reduced, the word $W_0\Delta_{11}^{-1}\Delta_2^k\Delta_{11}U_2$ is reduced and $||W_0\Delta_{11}^{-1}\Delta_2^k\Delta_{11}U_2|| = |W|$. Hence

$$\begin{aligned} ||\Delta^{-k}V\Delta^k|| &= ||U_1^{-1}U_2^{-1}\Delta_{11}^{-1}\Delta_2^{-k}\Delta_{11}W\Delta_{11}^{-1}\Delta_2^k\Delta_{11}U_2U_1|| \\ &= ||U_1^{-1}W_0\Delta_{11}^{-1}\Delta_2^k\Delta_{11}U_2U_1|| \\ &= |W| + 2|U_1| < |V|, \end{aligned}$$

because $|U_1| < |U|$.

Remark that with such V and Δ the inequality $||\Delta^{-1}V\Delta|| > |V|$ also holds. Indeed,

$$\begin{aligned} \Delta^{-1}V\Delta &\equiv U^{-1}\Delta_{11}^{-1}\Delta_2^{-1}\Delta_{11}W\Delta_{11}^{-1}\Delta_2\Delta_{11}U \\ &\equiv U_1^{-1}U_2^{-1}\Delta_{11}^{-1}\Delta_2^{-1}\Delta_{11}(\Delta_{11}^{-1}\Delta_2^k\Delta_{11}U_2W_0)\Delta_{11}^{-1}\Delta_2\Delta_{11}U_2U_1 \\ &= U_1^{-1}U_2^{-1}\Delta_{11}^{-1}\Delta_2^{(k-1)}\Delta_{11}U_2W_0\Delta_{11}^{-1}\Delta_2\Delta_{11}U_2U_1. \end{aligned}$$

Since the obtained word is reduced, we get $||\Delta^{-1}V\Delta|| = |W| + 2|U| + 2|\Delta_{11}|$, but $|V| = |W| + 2|U|$.

Also, it is easy to see that $||\Delta V\Delta^{-1}|| > |V|$.

Thus we get that if

$$\Delta \equiv U_1^{-1}U_2^{-1}\Delta_{11}^{-1}\Delta_2\Delta_{11}U_2U_1$$

and

$$V \equiv U_1^{-1}U_2^{-1}\Delta_{11}^{-1}\Delta_2^k\Delta_{11}U_2W_0U_2U_1$$

for some $\Delta_{11}, \Delta_2, U_1, U_2, W_0 \in F_n$, and some integer $k > 1$, then $||\Delta^{-\varepsilon}V\Delta^{\varepsilon}|| > |V|$, but $||\Delta^{-k}V\Delta^k|| < |V|$. Note that we have $W = \Delta_{11}^{-1}\Delta_2^k\Delta_{11}U_2W_0$, that gives the case (2a) of the statement for the case $m = 0$.

Above we considered the case when in the product $U_2^{-1}\Delta_{11}^{-1}\Delta_2^{-k}\Delta_{11}W$ the word $U_2^{-1}\Delta_{11}^{-1}\Delta_2^{-k}\Delta_{11}$ was cancelling wholly with initial part of W of with the whole W, but without further cancellations. Now, let us consider the case when W is cancelling wholly and after that there are further cancellations in the product

$$\Delta^{-k}V\Delta^k = U_1^{-1}U_2^{-1}\Delta_{11}^{-1}\Delta_2^{-k}\Delta_{11}W\Delta_{11}^{-1}\Delta_2^k\Delta_{11}U_2U_1.$$

Remark, that it is possible only if the word $U_2^{-1}\Delta_{11}^{-1}\Delta_2^{-k}\Delta_{11}$ is a product of some initial subword Φ and of an element from the centralizer of W, i.e.

$$U_2^{-1}\Delta_{11}^{-1}\Delta_2^{-k}\Delta_{11} \equiv \Phi W^{-m}, \quad m \in \mathbb{N}$$

In this case we assume that m is maximal integer with such property, i.e. Φ does not contain word W^{-1} as a final part. Then

$$\Delta^{-k}V\Delta^k = U_1^{-1}\Phi W^{-m}WW^m\Phi^{-1}U_1 = U_1^{-1}\Phi W\Phi^{-1}U_1$$

and after that Φ is cancelling with an initial subword of W, i.e. $W \equiv \Phi^{-1}W_0$, where W_0 can, possibly, be empty. If W_0 is empty, then, obviously, $U_1^{-1}\Phi^{-1}U_1$ must be reduced.

Case 2(iv). Suppose that $|U| < |\Delta_1|$ and cancellations in W starts from the final part. By the same arguments as in the Case 2(iii), we get that words Δ and V are of the form, described in case (2b).

□

If Δ is not cyclically reduced, Lemma 4 gives the finite algorithm to find for a given reduced word V a Δ-reduced word V_Δ conjugated to V by some power of Δ. If Δ and V are of the form represented in case (2) of Lemma 4, then we define $V_\Delta = \Delta^{-k} V \Delta^k$. In this case we say that V_Δ is obtained from V by Δ-reducing. If Δ and V are others, then we follow the same steps as described after Lemma 3.

5. Δ-reduced words corresponding to the same word

In general, V_Δ is not uniquely determined by V. The following statement describes different Δ-reduced words corresponding to the same reduced word.

Proposition 2. *(1) Let V be reduced and $|V| = ||\Delta^{-1} V \Delta||$. Then*

$$\Delta \equiv \Delta_1 \Delta_{31} \Delta_{21} \Delta_{31} \Delta_{32},$$

where $|\Delta_1| = |\Delta_{32}|$, with either

$$V \equiv \Delta_1 V_0 \Delta_{21}^{-1} \Delta_{31}^{-1} \Delta_1^{-1} \quad or \quad V \equiv \Delta_1 \Delta_{31} \Delta_{21} V_0 \Delta_1^{-1},$$

where $\Delta_{31} \equiv 1$ if $V_0 \neq 1$, for some $\Delta_1, \Delta_{21}, \Delta_{31}, \Delta_{32}, V_0 \in F_n$.
(2) Let V be reduced and $|V| = ||\Delta V \Delta^{-1}||$. Then

$$\Delta \equiv \Delta_{11} \Delta_{12} \Delta_{21} \Delta_{12} \Delta_3,$$

where $|\Delta_{11}| = |\Delta_3|$, with either

$$V \equiv \Delta_3^{-1} V_0 \Delta_{21} \Delta_{12} \Delta_3 \quad or \quad V \equiv \Delta_3^{-1} \Delta_{12}^{-1} \Delta_{21}^{-1} V_0 \Delta_3,$$

where $\Delta_{12} \equiv 1$ if $V_0 \neq 1$, for some $\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_3, V_0 \in F_n$.

Proof. Let us prove the statement (1). Represent a Δ-reduced word V in the form $V \equiv U^{-1} W U$, where W is cyclically reduced and U is reduced (we assume that U is empty if V is cyclically reduced itself, i.e. $V \equiv W$). Since $|V| = ||\Delta V \Delta^{-1}||$, the word $\Delta^{-1} V \Delta \equiv \Delta^{-1} U^{-1} W U \Delta$ admits cancellations. There are two possibilities: either there are cancellations with letters of W or not.

Case 1: Suppose that there are cancellations of letters of Δ with letters of U, where, possibly, U will be cancelled wholly (the same for Δ^{-1} and U^{-1}, respectively), but there no further cancellations with letters of W. Then $U \equiv U_1 \Delta_1^{-1}$ and $\Delta \equiv \Delta_1 \Delta_3$, where the reduced word Δ_1 is non-empty and reduced words U_1 and Δ_3 are, possibly, empty. Therefore,

$$\Delta^{-1} V \Delta \equiv \Delta_3^{-1} \Delta_1^{-1} (\Delta_1 U_1^{-1} W U_1 \Delta_1^{-1}) \Delta_1 \Delta_3 = \Delta_3^{-1} U_1^{-1} W U_1 \Delta_3$$

where the final word is reduced. So,

$$||\Delta^{-1} V \Delta|| = |W| + 2|U_1| + 2|\Delta_3|.$$

Since
$$|V| = |\Delta_1 U_1^{-1} W U_1 \Delta_1^{-1}| = |W| + 2|U_1| + 2|\Delta_1|,$$
we get $|\Delta_1| = |\Delta_3|$. Taking $V_0 \equiv U_1^{-1} W U_1$, $\Delta_{32} \equiv \Delta_3$, $\Delta_{21} \equiv 1$ and $\Delta_{31} \equiv 1$ we get Δ and V of the same form as in the statement.

Case 2: Suppose that words U and U^{-1} will be cancelled wholly and there are further cancellations with letters of W, starting either from the initial part of W or from the final part of W, but not from both, because W is cyclically reduced.

Case 2(i): Suppose that there are cancellations starting from the final part of W such that, possibly, W cancelled wholly, but there are no further cancellations. Then $\Delta \equiv \Delta_1 \Delta_2 \Delta_3$, $U \equiv \Delta_1^{-1}$ and $W \equiv V_0 \Delta_2^{-1}$ for some reduced $\Delta_1, \Delta_2, \Delta_3, V_0 \in F_n$. Therefore, $V \equiv \Delta_1 V_0 \Delta_2^{-1} \Delta_1^{-1}$ and $\Delta^{-1} V \Delta = \Delta_3^{-1} \Delta_2^{-1} V_0 \Delta_3$. Since
$$|V| = |V_0| + |\Delta_2| + 2|\Delta_1|$$
and
$$||\Delta^{-1} V \Delta|| = |V_0| + |\Delta_2| + 2|\Delta_3|,$$
we get $|\Delta_1| = |\Delta_3|$. Taking $\Delta_{21} \equiv \Delta_2$, $\Delta_{32} \equiv \Delta_3$ and $\Delta_{31} \equiv 1$ we get Δ and V of the same form as in the statement (with $V_0 \equiv 1$ if W cancelled wholly).

Case 2(ii): Suppose that W is cancelled wholly starting from the final part and after that some cancellations are still possible. Therefore, $V_0 \equiv 1$, $\Delta^{-1} V \Delta \equiv \Delta_3^{-1} \Delta_2^{-1} \Delta_3$ and some cancellations of letters of Δ_2^{-1} and Δ_3 are possible. Thus, denoting by Δ_{32} the part (possibly, empty) of Δ_3 which cannot be cancelled, we can write $\Delta_3 \equiv \Delta_{31} \Delta_{32}$ and $\Delta^{-1} V \Delta = \Delta_{32}^{-1} \Delta_{31}^{-1} \Delta_2^{-1} \Delta_{31} \Delta_{32}$, where $\Delta_{31}^{-1} \Delta_2^{-1} \Delta_{31}$, after possible cancellations, is cyclically reduced. So, elements Δ_2^{-1} and $\Delta_{31}^{-1} \Delta_2^{-1} \Delta_{31}$ are conjugated in the free group and cyclically reduced. Therefore, the reduction of the word $\Delta_{31}^{-1} \Delta_2^{-1} \Delta_{31}$ is a cyclic permutation of the word Δ_2^{-1}, so $\Delta_2^{-1} = \Delta_{21}^{-1} \Delta_{31}^{-1}$ and $\Delta_{31}^{-1} \Delta_2^{-1} \Delta_{31}^{-1} = \Delta_{31}^{-1} \Delta_{21}^{-1} \Delta_{31}^{-1} \Delta_{31} = \Delta_{31}^{-1} \Delta_{21}^{-1}$. Hence
$$\Delta \equiv \Delta_1 \Delta_2 \Delta_3 \equiv \Delta_1 \Delta_{31} \Delta_{21} \Delta_{31} \Delta_{32},$$
$$V \equiv \Delta_1 \Delta_2^{-1} \Delta_1^{-1} \equiv \Delta_1 \Delta_{21}^{-1} \Delta_{31}^{-1} \Delta_1^{-1}$$
and
$$\Delta^{-1} V \Delta = \Delta_3^{-1} \Delta_2^{-1} \Delta_3 = \Delta_{32}^{-1} \Delta_{31}^{-1} \Delta_{21}^{-1} \Delta_{31}^{-1} \Delta_{31} \Delta_{32} = \Delta_{32}^{-1} \Delta_{31}^{-1} \Delta_{21}^{-1} \Delta_{32}.$$
Since
$$|V| = 2|\Delta_1| + |\Delta_{21}| + |\Delta_{31}|$$
and
$$||\Delta^{-1} V \Delta|| = 2|\Delta_{32}| + |\Delta_{21}| + |\Delta_{31}|,$$
we get $|\Delta_1| = |\Delta_{32}|$. Taking $V_0 \equiv 1$ we get that Δ and V are of the same form as in the statement.

Case 2(iii): Suppose that there are cancellations starting from the initial part of W such that, possibly, W cancelled wholly, but there no further cancellations. Then $\Delta \equiv \Delta_1\Delta_2\Delta_3$, $U \equiv \Delta_1^{-1}$ and $W \equiv \Delta_2 V_0$ for some reduced words $\Delta_1, \Delta_2, \Delta_3, V_0 \in F_n$. Therefore, $V \equiv \Delta_1\Delta_2 V_0 \Delta_1^{-1}$ and $\Delta^{-1}V\Delta = \Delta_3^{-1}V_0\Delta_2\Delta_3$. Since

$$|V| = |V_0| + |\Delta_2| + 2|\Delta_1|$$

and

$$||\Delta^{-1}V\Delta|| = |V_0| + |\Delta_2| + 2|\Delta_3|,$$

we get $|\Delta_1| = |\Delta_3|$. Taking $\Delta_{21} \equiv \Delta_2$, $\Delta_{32} \equiv \Delta_3$ and $\Delta_{31} \equiv 1$ we get that Δ and V are of the form as in the statement.

Case 2(iv): Suppose that W is cancelled wholly (starting from the initial part) and after that some cancellations are still possible. Therefore, $V_0 \equiv 1$, $\Delta^{-1}V\Delta = \Delta_3^{-1}\Delta_2\Delta_3$ and some cancellations of letters of Δ_3^{-1} and Δ_2 are possible. Denoting by Δ_{32}^{-1} the part (possibly, empty) of Δ_3^{-1} which can not be cancelled, we can write $\Delta_3^{-1} \equiv \Delta_{32}^{-1}\Delta_{31}^{-1}$, so $\Delta_3 \equiv \Delta_{31}\Delta_{32}$. Therefore we get $\Delta^{-1}V\Delta = \Delta_3^{-1}\Delta_2\Delta_3 = \Delta_{32}^{-1}\Delta_{31}^{-1}\Delta_2\Delta_{31}\Delta_{32}$, where $\Delta_{31}^{-1}\Delta_2\Delta_{31}$, after possible cancellations, is cyclically reduced. Elements Δ_2 and $\Delta_{31}^{-1}\Delta_2\Delta_{31}$ are conjugated in the free group and cyclically reduced. Therefore, the reduction of the word $\Delta_{31}^{-1}\Delta_2\Delta_{31}$ is a cyclic permutation of the word Δ_2^{-1}, so $\Delta_2 = \Delta_{31}\Delta_{21}$ and $\Delta_{31}^{-1}\Delta_2\Delta_{31} = \Delta_{31}^{-1}\Delta_{31}\Delta_{21}\Delta_{31} = \Delta_{21}\Delta_{31}$. Hence

$$\Delta \equiv \Delta_1\Delta_2\Delta_3 \equiv \Delta_1\Delta_{31}\Delta_{21}\Delta_{31}\Delta_{23},$$
$$V \equiv \Delta_1\Delta_2\Delta_1^{-1} \equiv \Delta_1\Delta_{31}\Delta_{21}\Delta_1^{-1},$$

and

$$\Delta^{-1}V\Delta = \Delta_3^{-1}\Delta_2\Delta_3 = \Delta_{32}^{-1}\Delta_{31}^{-1}\Delta_{31}\Delta_{21}\Delta_{31}\Delta_{32} = \Delta_{32}^{-1}\Delta_{21}\Delta_{31}\Delta_{32}.$$

Since

$$|V| = 2|\Delta_1| + |\Delta_{21}| + |\Delta_{31}|$$

and

$$||\Delta^{-1}V\Delta|| = 2|\Delta_{32}| + |\Delta_{21}| + |\Delta_{31}|,$$

we get $|\Delta_1| = |\Delta_{32}|$. Taking $V_0 \equiv 1$ we get that Δ and V are of the form as in the statement.

The statement (2) follows by similar considerations. $\square$

6. Constructing of φ-twisted conjugated normal form

Let V and V' be words in the alphabet $\mathbb{X}$. If there exists $X \in F_n$ such that $V' = \varphi(X^{-1})VX$, we say that V' and V are *φ-twisted conjugated*, and that V' is obtained from V by *φ-twisted conjugation by X*. Obviously, the property to be φ-twisted conjugated is equivalence relation in the group F_n. Using this definition, the Makanin's question can be reformulated as the following: is there exists an algorithm that admits for a given pair of elements U and V of a free group F_n to decide if they are φ-twisted conjugated.

In virtue of Lemma 2 the equality $\varphi(\Delta^k) = \Delta^k$ holds for any integer k. Therefore, if elements U and V are conjugated by some power of Δ, they are φ-twisted conjugated.

Below we will construct an algorithm to choose a unique representative for each class of φ-twisted conjugated elements. We will call this representative the *φ-twisted conjugated normal form*. It will be shown that two elements of F_n are φ-twisted conjugated if and only if their φ-twisted conjugated normal forms coincide.

If length of a reduced word U in the alphabet $\mathbb{X}$ is bigger than 1 then it can be represented as a product $U \equiv U'U''$ of two nonempty reduced words. The word U' will be referred to as an *initial part* of U and U'' will be referred as a *final part* of U. Denote by $I(U)$ the set of all initial parts of U and by $F(U)$ the set of all final parts of U.

For a word U and an integer ℓ we denote by $\varphi^{\ell}(U)$ the word obtained from U by replacing (graphically) each letter u of U by its image $\varphi^{\ell}(u)$, and denote by $U_{[\ell]}$ the word obtained after reducing of $\varphi^{\ell}(U)$.

Let $V \equiv V'V''$ be a reduced word in the alphabet $\mathbb{X}$, where $V' \in I(V)$, $V'' \in F(V)$. The word $V''_{[1]}V'$ (that represents the element $\varphi(V'')V(V'')^{-1}$ is said to be a *cyclic φ-shift of a final part of* $V \equiv V'V''$ and the word $V''V'_{[-1]}$, that represents the element

$$\varphi([\varphi^{-1}(V')]^{-1})V\varphi^{-1}(V') = (V')^{-1}V\varphi^{-1}(V'),$$

is said to be a *cyclic φ-shift of an initial part of* $V \equiv V'V''$. If $|V'| = 1$, i.e. $V' = x_i^{\varepsilon}$, $\varepsilon = \pm 1$, then the corresponding φ-shift of the initial part will be referred to as a *cyclic φ-shift of the initial letter*. If $|V''| = 1$ then the corresponding φ-shift of the final part will be referred to as a *cyclic φ-shift of the final letter*.

A reduced word V will be referred to as a *cyclically φ-reduced* if neither a cyclic φ-shift of its final letter nor a cyclic φ-shift of its initial letter do not decrease length of V. Obviously, applying cyclic φ-shifts of the final letter (as well as of the initial letter) to a given word V, after a finite number of steps we will obtain a cyclically φ-reduced word corresponding to V.

Now let us construct conjugated normal form for a word V. Without loss of generality (applying, if necessary, finite number of steps of above described Δ-reducings and cyclic φ-shifts), we can assume that V is Δ-reduced and cyclically φ-reduced.

For given V let $\mathcal{V}_{\Delta}$ be the set, consisting of V and all words which are conjugated to V by powers of Δ and are Δ-reduced. In virtue of Lemma 3 and Lemma 4, the set $\mathcal{V}_{\Delta}$ is finite. Applying to all elements of $\mathcal{V}_{\Delta}$ cyclic φ-shifts of all initial parts and all final parts, we will construct the set $(\mathcal{V}_{\Delta})_{\varphi}$. Applying Δ-reducing to all element of $(\mathcal{V}_{\Delta})_{\varphi}$, we will construct the set of Δ-reduced words $((\mathcal{V}_{\Delta})_{\varphi})_{\Delta}$. And finally, applying, if necessary, cyclic φ-shifts of the final letter and of the initial letter to words from $((\mathcal{V}_{\Delta})_{\varphi})_{\Delta}$, will construct the set $D_0(V) = (((\mathcal{V}_{\Delta})_{\varphi})_{\Delta})_{\varphi}$ of Δ-reduced and cyclically φ-reduced words.

Recall that in the free group the set of words obtained from a given word V by cyclic shifts is finite. But the set of words obtained from V by cyclic φ-shifts can be infinite. Indeed, it is clear from relations

$$\left[\varphi^k(V)\varphi^{k-1}(V)\cdots\varphi(V)\right] V \left[V^{-1}\,\varphi(V^{-1})\cdots\varphi^{k-1}(V^{-1})\right] = \varphi^k(V),$$

for $k > 0$ and

$$\left[\varphi^k(V^{-1})\varphi^{k+1}(V^{-1})\cdots V^{-1}\right] V \left[\varphi^{-1}(V)\cdots\varphi^{k+1}(V)\varphi^k(V)\right] = \varphi^k(V),$$

for $k < 0$, that V is φ-twisted conjugated to $\varphi^k(V)$ for any integer k.

For each integer k we define a set $D_k(V) = D_0(\varphi^k(V))$ and define $D(V) = \bigcup_{k\in\mathbb{Z}} D_k(V)$. Let us verify that $D(V)$ is finite.

Lemma 5. *The following equality holds:*

$$D(V) \;=\; \bigcup_{k\in\{0,1,\ldots,m-1\}} D_k(V).$$

Proof. Let r, satisfying $0 \le r < m$, be such that $k = mq + r$, $q \in \mathbb{Z}$. By Lemma 2,

$$\varphi^k(V) \;=\; \Delta^{-q}\,\varphi^r(V)\,\Delta^q.$$

We show that $D_k(V) = D_r(V)$, that will imply the statement. Indeed, by the definition, $D_k(V) = D_0(\Delta^{-q}\varphi^r(V)\Delta^q)$. Denote $U \equiv V_{[r]} = \varphi^r(V)$ and consider elements from $D_0(U) = D_0(\varphi^r(V))$. By the definition, $D_0(U)$ consists of words $\varphi(U'')U'$ and $U''\varphi^{-1}(U')$ (for all pairs of initial and final parts of $U \equiv U'U''$) to which Δ-reducing and cyclic φ-reducing are applied. To construct $D_k(V)$ we need to pass from the word $\Delta^{-q}\varphi^r(V)\Delta^q$ to a Δ-reduced word, which, in a general case, can be different from $U \equiv \varphi^r(V)$. But, according to the definition of $D_0(V)$, the set of Δ-reduced words constructed from $\Delta^{-q}\varphi^r(V)\Delta^q$ coincides with the set of Δ-reduced words constructed from $\varphi^r(V)$. So, corresponding sets of all cyclic φ-shifts of initial parts and of final parts also coincide. Therefore, $D_k(V) = D_0(\varphi^r(V)) = D_r(V)$. $\square$

Lemma 6. *The set $D(V)$ has the following properties:*
(1) If $U \in D(V)$ then $D(U) = D(V)$;
(2) If V and W are φ-twisted conjugated, and each of them is Δ-reduced and cyclically φ-reduced, then $D(V) = D(W)$.

Proof. Since $D(V)$ consists of words which are φ-twisted conjugated in F_n, item (2), obviously, implies (1). Let us prove (2). Firstly we remark that if W is obtained from V by a φ-shift of an initial part or a φ-shift of a final part, then, obviously, $D(W) = D(V)$. Now assume that W and V are related by $W = U_{[1]}^{-1}VU$ for some reduced $U \in F_n$. Since W is cyclically φ-reduced, the product $U_{[1]}^{-1}VU$ contains cancellations. Moreover, these cancellations are either in the product $U_{[1]}^{-1}V$ or in the product VU, but not in the both, since by the assumption V is φ-reduced. There are two possibilities: V will be cancelled wholly or not.

Case 1: Suppose V is not cancelling wholly.

Case 1(i): Suppose that there are cancellations in the product $U_{[1]}^{-1}V$. Then $U_{[1]}^{-1}$ must be cancelled wholly, since W is assumed to be φ–reduced, and so $V \equiv U_{[1]}V_1$. Then

$$W = U_{[1]}^{-1}VU \equiv U_{[1]}^{-1}(U_{[1]}V_1)U = V_1U$$

and $W \equiv V_1U$ is obtained from $V \equiv U_{[1]}V_1$ by the φ-shift of the initial part $U_{[1]}$. Therefore, $D(V) = D(W)$.

Case 1(ii): Suppose that there are cancellations in the product VU. Then U must be cancelled wholly since W is assumed to be φ–reduced, and so $V \equiv V_1U^{-1}$. Then

$$W = U_{[1]}^{-1}VU \equiv U_{[1]}^{-1}(V_1U^{-1})U = U_{[1]}^{-1}V_1$$

and $W \equiv U_{[1]}^{-1}V_1$ is obtained from $V \equiv V_1U^{-1}$ by the φ-shift of the final part U^{-1}. Therefore, $D(V) = D(W)$.

Case 2: Suppose that V is cancelling wholly. We will use induction by length of the word U.

Case 2(i): Suppose that V is cancelling wholly in the product VU and after that there are cancellations of letters of the remaining part of U with letters of $U_{[1]}^{-1}$. Then we can represent $U \equiv V^{-1}U_1$, therefore $U_{[1]}^{-1} = U_{1\,[1]}^{-1}V_{[1]}$ and

$$U_{[1]}^{-1}VU = U_{1\,[1]}^{-1}V_{[1]}VV^{-1}U_1 = U_{1\,[1]}^{-1}V_{[1]}U_1.$$

If U_1 is cancelling wholly with $V_{[1]}$, then $V_{[1]} \equiv V_2U_1^{-1}$ and

$$U_{1\,[1]}^{-1}V_{[1]}U_1 \equiv U_{1\,[1]}^{-1}V_2U_1^{-1}U_1 = U_{1\,[1]}^{-1}V_2 = W.$$

Remark that W arises in the process of the construction of $D_1(V)$. Indeed,

$$D_1(V) = D_0(V_{[1]}) = D_0(V_2U_1^{-1})$$

and W is obtained from $V_2U_1^{-1}$ by the cyclic φ-shift of the final part. If, again, $V_{[1]}$ is cancelling wholly with U_1, then the statement follows from the induction assumption.

Case 2(ii): Let there be a cancellation in the product $U_{[1]}^{-1}V$ with V cancelling wholly and remaining part of $U_{[1]}^{-1}$ cancelling with U, i.e. $U_{[1]}^{-1} \equiv U_1^{-1}V^{-1}$, $U = \varphi^{-1}(V)\varphi^{-1}(U_1)$ and

$$U_{[1]}^{-1}VU = U_1^{-1}V^{-1}VV_{[-1]}U_{1\,[-1]} = U_1^{-1}V_{[-1]}U_{1\,[-1]}. \tag{1}$$

If U_1^{-1} is cancelling wholly with $V_{[-1]}$, then $V_{[-1]} = U_1V_2$ and we get $U_1^{-1}V_{[-1]}U_{1\,[-1]} = U_1^{-1}(U_1V_2)U_{1\,[-1]} = V_2U_{1\,[-1]}$, i.e. $V = U_{1\,[1]}V_{2\,[1]}$ and $W \equiv V_2U_{1[-1]}$. Comparing these words we see that W belongs to $D_{-1}(V)$. Indeed, by the definition,

$$D_{-1}(V) = D_0(\varphi^{-1}(V)) = D_0(U_1V_2).$$

Applying to U_1V_2 the cyclic φ-shift of the initial part U_1, we will get $W = V_2U_{1\,[-1]}$. The case when $V_{[-1]}$ is cancelling wholly with U_1^{-1} in (1) follows from the induction assumption. □

Now we are able to complete the proof of Theorem 1.

Proof. For given V we have constructed the finite set $D(V)$. From this set we choose words of minimal length, and after that, from such words choose the word which is minimal in respect to the above defined ordering on $\mathbb{F}_n$. Denote this word by $\overline{V}$ and call it the *normal φ-twisted conjugated form for V*.

Let U and V be reduced words in the free group F_n. Let us construct words $\overline{U}$ and $\overline{V}$, which are normal φ-conjugated forms for U and V, respectively. In virtue of Lemma 5, $\overline{U} = \overline{V}$ if and only if elements U and V are φ-twisted conjugated in F_n, that means, by the definition, that equation $\varphi(X)U = VX$ is solvable in F_n. The proof is completed. □

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